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Scattering amplitudes and the AdS/CFT correspondence

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Abstract
We give a pedagogical account of recent progress concerning the computation of scattering amplitudes of planar maximally supersymmetric Yang–Mills in four dimensions at strong coupling by means of the AdS/CFT correspondence. Two important ingredients of the computation are the presence of a 'dual' conformal symmetry and the equivalence between the scattering amplitude and a particular kind of light-like cusped Wilson loops. We discuss recent progress extending these two features to all values of the coupling constant.

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1. Introduction

In this review, we study gluon scattering amplitudes of planar maximally supersymmetric Yang–Mills (MSYM). Such amplitudes would teach us something about QCD amplitudes, but at the same time are much more tractable. The reason for such simplicity is twofold. On the one hand, perturbative computations are much simpler than in QCD, and in fact enormous progress has been made in the last few years. On the other hand, the strong coupling regime of the theory can be studied by means of the AdS/CFT duality, by studying a weakly coupled string sigma model.

The aim of these lectures is to explain how to use the AdS/CFT duality in order to compute gluon scattering amplitudes of planar MSYM at strong coupling [1, 2].

In the following section, we briefly describe some weak-coupling perturbative results for scattering amplitudes in planar MSYM. Scattering amplitudes of massless gauge theories in four dimensions suffer from IR divergences; hence, a regulator needs to be introduced in order to define them properly. A convenient choice is (a version of) dimensional regularization, in which the theory is studied in $D = 4 - 2\epsilon$ dimensions.

In section 3, we proceed and explain how the AdS/CFT duality can be used to compute scattering amplitudes at strong coupling. As in the weak-coupling computation, amplitudes are IR divergent and a regulator has to be introduced. In order to set up the computation we introduce a D-brane as a regulator. Actual computations, however, are done by using the analogous of dimensional regularization, since we are interested in comparing our results to the expectations from the perturbative regime. We show in detail how the prescription works for the scattering of four gluons. An important ingredient of the computation is the presence of a novel 'dual' conformal symmetry. We briefly discuss how such a symmetry constrains the amplitudes. At the end of section 3, we briefly discuss recent development regarding the extension of dual conformal symmetry to a larger dual super-conformal symmetry.

In section 4, we review a recent conjectured duality between a particular class of scattering amplitudes (the so-called maximally helicity violating (MHV) amplitudes) and cusped Wilson loops. The one-loop duality, proven for the generic case of n-gluons, can be combined with our prescription, described in section 3, in order to test a particular guess for the form of the amplitudes.

Finally, in section 5 we end up with some conclusions and a list of open problems. For a more detailed exposition of the subject of these lectures, we refer the reader to [3].

2. Perturbative scattering amplitudes of planar MSYM

In this section, we briefly discuss the progress during the last few years in computing perturbative planar scattering amplitudes of MSYM.

Gluon states $|\mathcal{G}\rangle = |h, p^\mu, a\rangle$ are characterized by their helicity, $h = \pm 1$, four momenta, p^μ, with $p^2 = 0$ and color indices a in the adjoint representation of the gauge group $SU(N)$. Generic amplitudes depend in a complicated manner on these.

In the planar limit it is useful to write the amplitudes in the so-called color decomposed form:

$$\mathcal{A}_n^{(L)} \sim g^{n-2}(g^2N)^L \sum_{\rho} \mathrm{Tr}(T^{a_{\rho(1)}} \cdots T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), \ldots, \rho(n)), \tag{2.1}$$

where we denoted by $\mathcal{A}_n^{(L)}$ the L-loop, n-point amplitude. The sum runs over non-cyclic permutations, N denotes the number of colors and g the gauge theory coupling constant. In the nonplanar case, there are also contributions with multi-traces, suppressed in the large N limit. This decomposition separates the color structure from the kinematics. The leading color ordered amplitude $A_n^{(L)}$ will only depend on the momenta and the helicities of the particles undergoing the scattering.

As already mentioned, scattering amplitudes of massless particles in four dimensions suffer from infrared divergences. In order to see this more clearly, consider the scattering of four gluons at one loop. The amplitude contains a factor of the form

$$A_{4,D=4}^{(1)} \sim \int \frac{\mathrm{d}^4 p}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}, \tag{2.2}$$

where p is the momentum running along the loop, and $k_i^2 = 0$ are the momenta of the external particles. We recognize two kinds of divergences. First, from the region $p^\mu \sim 0$. These are called soft divergences, since they are due to the interchange of gluons with very low momenta (soft) among external gluons. The second class of divergences comes from the region $p^\mu \sim \alpha k_i^\mu$ and are called collinear divergences, since in this case the momentum of the gluon interchanged is parallel to the momentum of one of the external gluons. Note that, in addition, we can have $p^\mu \sim \alpha k_i^\mu$ with $\alpha \sim 0$, so soft and collinear effects can combine, giving an enhanced divergence.

Such amplitude can be regularized in several ways; for instance, we could put the external particles off-shell, so that $k_i^2 = m_i^2$. However, it has been proven more convenient to use a version of dimensional regularization, and consider the theory in $D = 4 - 2\epsilon$ dimensions. More precisely, we still consider four-dimensional external states but consider the momentum running inside the loop in a dimension slightly higher than four (ϵ is negative):

$$A^{(1)}_{4,D=4} \to A^{(1)}_{4,D=4-2\epsilon}. \tag{2.3}$$

Once a regulator is introduced, the amplitudes are finite, but the price we have to pay is an explicit dependence on such a regulator[1]. In the case of dimensional regularization, IR divergences manifest themselves as poles in ϵ. From the one-loop expression (2.2) and the explanation above, we see that soft and collinear divergences combine to give a divergent factor $\approx\frac{1}{\epsilon^2}$. At L loops we expect a divergence of the form

$$A^{(L)}_n \sim \frac{1}{\epsilon^{2L}} + \cdots. \tag{2.4}$$

As already mentioned, the color ordered amplitudes depend also on the helicities of the external gluons. For instance, it can be shown that tree-level amplitudes vanish if all helicities, or all but one, are plus:

$$\mathcal{A}(++\cdots+) = \mathcal{A}(-+\cdots+) = 0. \tag{2.5}$$

As a consequence of supersymmetry, scattering amplitudes of MSYM satisfy supersymmetric Ward identities. In particular, relations (2.5) are a consequence of such identities and as a result, valid to all values of the coupling constant. The first non-trivial amplitude is the so-called MHV amplitude, $\mathcal{A}(--++\cdots+)$, in which two external states have minus helicity and the rest plus. A great simplicity of studying MHV amplitudes is given by the fact that they contain a single Lorentz structure, which is already captured by the tree-level amplitude, to all orders in perturbation theory. It is then convenient to factor out the tree-level amplitude and study the so-called reduced amplitude

$$M^{(L)}_n(\epsilon) = \frac{A^{(L)}_n(\epsilon)}{A^{(0)}_n}, \tag{2.6}$$

which depends only on the kinematical invariants and the regulator.

In the last few years, technology has been developed for the computation of perturbative scattering amplitudes; see, for instance, [4]. Such technology made possible the computation of scattering amplitudes for four (and to some extent for five) gluons to rather high order in the loop expansion. Bern, Dixon and Smirnov (BDS) noticed in [5], after previous hints in [6], that all known MHV amplitudes satisfied a recursion relation, allowing us to write amplitudes at a given order in terms of lower order amplitudes. In particular, the following ansatz was proposed for MHV amplitudes of any number of gluons at any loop order:

$$\mathcal{M} \equiv 1 + \sum_{\ell=1} \alpha^\ell M^{(\ell)}_n = \exp\left[\sum_{\ell=1}\left(f^{(\ell)}(\epsilon) M^1_n(\ell\epsilon) + C^\ell + \mathcal{O}(\epsilon)\right)\right], \tag{2.7}$$

where

$$f^{(\ell)}(\epsilon) = f^{(\ell)}_0 + \epsilon f^{(\ell)}_1 + \epsilon^2 f^{(\ell)}_2, \qquad \alpha \approx \lambda\mu^{2\epsilon}, \tag{2.8}$$

where α is proportional to the 't Hooft coupling constant and keeps track of the perturbation order. We have also introduced a IR scale μ, since the coupling constant in dimensions different from four is dimensionfull.

The above relation, from now on called the BDS ansatz, expresses the (log of the) amplitude in terms of the one-loop amplitude $M^1_n(\ell\epsilon)$. Note that this relation is highly non-trivial, since the constants $f^{(\ell)}_i$ and C^ℓ do not depend on the number of gluons or the kinematics. Note that in addition to the constant piece, there can be non-iterating terms of order $\mathcal{O}(\epsilon)$.

[1] And the consequent breaking of some of the symmetries present in the original problem.

The structure of the IR divergent terms in (2.7) was determined in [7], where it was shown that indeed IR divergences factorize and exponentiate as in (2.7). The highly non-trivial content of the ansatz is that the finite pieces also exponentiate in a similar manner and are given, at all loops, by the same functions that characterize the IR divergent terms (times the finite pieces at one loop).

We will be particularly interested in the scattering of four gluons; in that case the BDS ansatz reduces to

$$A_4 = A_{\text{tree}}(A_{\text{div},s})^2(A_{\text{div},t})^2 \exp\left(\frac{f(\lambda)}{8}\left(\log\frac{s}{t}\right)^2 + \text{const}\right), \tag{2.9}$$

$$A_{\text{div}_s} = \exp\left(-\frac{1}{8\epsilon^2}f^{(-2)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon}g^{(-1)}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right), \tag{2.10}$$

where s and t are the usual Mandelstan variables for the scattering of four particles. The amplitude has two pieces, a divergent one and a finite one, and is characterized by two functions. The so-called cusp anomalous dimension, $f(\lambda) = (\lambda\partial_\lambda)^2 f^{(-2)}(\lambda)$, controls both the leading divergent piece and the finite piece, while the so-called collinear anomalous dimension, $g(\lambda)$, controls the subleading divergent pole.

The BDS ansatz satisfies a number of non-trivial tests:

- It agrees with explicit computations for four gluons at three loops and five gluons at two loops.
- As already mentioned, it is consistent with the well-known exponential structure of IR divergences.
- It has the correct collinear factorization properties, i.e. the correct limit when two adjacent external momenta become collinear.

At the moment, there are strong reasons to believe that the BDS ansatz is correct for four and five gluons, while direct evidence against it was recently found by an explicit computation for the scattering of six gluons at two loops [8].

Before proceeding, let us mention that the cusp anomalous dimension $f(\lambda)$ plays a very important role in recent developments regarding integrability in the AdS/CFT correspondence. It turns out that this function arises in many computations. In particular, it controls the large spin limit of the anomalous dimension of the so-called twist two operators, of the form $\text{Tr}\phi D^S\phi$, namely

$$\Delta - S = f(\lambda)\log S + \cdots \tag{2.11}$$

for large S, to all orders in perturbation theory! Using the integrability of the gauge theory dilatation operators [9] constructed an integral equation whose solution gives the cusp anomalous dimension to all values of the coupling constant. For instance, the equation can be solved at weak and strong values of the coupling [10–12], giving

$$f(\lambda) = \frac{\lambda}{2\pi^2} + \cdots, \qquad \lambda \ll 1, \qquad f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \cdots, \qquad \lambda \gg 1. \tag{2.12}$$

Such values agree with previous computations, done either at weak coupling, by using perturbation theory [13] or at strong coupling by using the AdS/CFT duality [19]. A much more detailed account of these developments is given in other contributions of the present issue.

In the following section, we will explain how to compute gluon scattering amplitudes of planar MSYM at strong coupling by using the AdS/CFT duality. In particular, for the case of four gluons, the strong-coupling results are in perfect agreement with (2.7).

3. Gluon scattering amplitudes at strong coupling

In order to attack the problem of computing scattering amplitudes at strong coupling we will make use of the AdS/CFT duality [15]. This duality expresses the equivalence between four-dimensional MSYM and type IIB string theory on $AdS_5 \times S^5$, and is one of the main subjects of the present issue.

In the limit of a large number of colors and large λ, the strings are well described by a weakly coupled sigma model.

As in the gauge theory, we need to introduce a regulator in order to define properly scattering amplitudes. In order to set up our computation we introduce a D-brane as an IR regulator. Actual computations, however, will be done by using the super-gravity analogous of dimensional regularization, since it is simpler to proceed in this scheme and besides we are interested in comparing our results to gauge theory expectations.

3.1. Set-up of the computation

As a first IR regulator we consider a D-brane localized in the radial direction. In other words, we start with the AdS_5 metric written in Poincaré coordinates,

$$\mathrm{d}s^2 = R^2 \frac{\mathrm{d}x_{3+1}^2 + \mathrm{d}z^2}{z^2}, \tag{3.1}$$

and we place a D-brane at some fixed large value of $z = z_{\mathrm{IR}}$ and extending along the x_{3+1} coordinates. The asymptotic states are open strings that end on that D-brane. We then consider the scattering of these open strings.

The proper momentum of the strings is $k_{\mathrm{pr}} = k z_{\mathrm{IR}}/R$, where k is the momentum conjugate to x_{3+1}, plays the role of gauge theory momentum and will be kept fixed as we take away the IR cut-off, $z_{\mathrm{IR}} \to \infty$. Therefore, due to the warping of the metric, the proper momentum is very large; so we are considering the scattering of strings at a fixed angle with very large momentum.

Amplitudes in such a regime were studied in flat space by Gross and Mende [16]. The key feature of their computation is that the amplitude is dominated by a saddle point of the classical action. In our case, we need to consider classical strings on AdS. Hence, we need to consider a world-sheet with the topology of a disk with vertex operator insertions on its boundary, which correspond to the external states (see figure 1). A disk amplitude with a fixed ordering of the open string vertex operators corresponds to a given color ordered amplitude.

The world-sheet is such that in the vicinity of a vertex operator, the momentum of the external state fixes the form of the solution (since, far away, the solution looks like a free string propagating with a given momentum) and, as the open strings are attached to the D-brane, $z = z_{\mathrm{IR}}$ at the boundary.

In order to state more simply the boundary conditions for the world-sheet, it is convenient to describe the solution in terms of T-dual coordinates y^μ, defined as follows:

$$\mathrm{d}s^2 = w^2(z)\, \mathrm{d}x_\mu\, \mathrm{d}x^\mu \to \partial_\alpha y^\mu = \mathrm{i} w^2(z) \epsilon_{\alpha\beta} \partial_\beta x^\mu. \tag{3.2}$$

Note that we do not T-dualize along the radial direction. After defining $r = R^2/z$ we end up again with the AdS_5 metric:

$$\mathrm{d}s^2 = R^2 \frac{\mathrm{d}y_\mu\, \mathrm{d}y^\mu + \mathrm{d}r^2}{r^2}. \tag{3.3}$$

In terms of the (3.3) coordinates, the boundary of the world-sheet is located at $r = R^2/z_{\mathrm{IR}}$. Besides, the boundary conditions for the original coordinates x^μ, which are that they carry

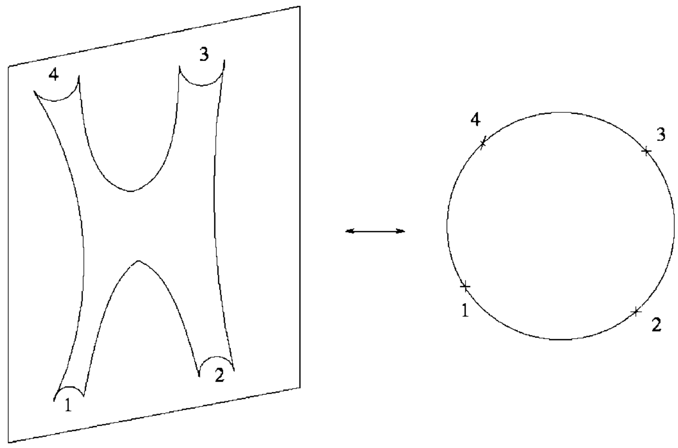

Figure 1. World-sheet corresponding to the scattering of four open strings. In the figure, on the left we see four open strings ending on the IR D-brane; the world-sheet has then the topology of a disk, shown on the right, with four vertex operator insertions.

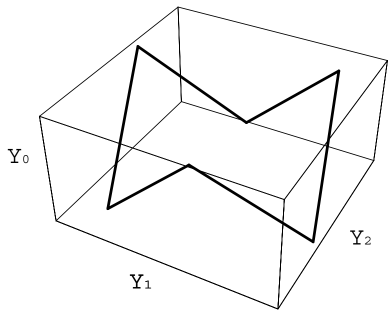

Figure 2. Polygon of light-like segments corresponding to the momenta of six external particles undergoing a scattering.

momentum k^μ, translate into the condition that y^μ has 'winding' $\Delta y^\mu = 2\pi k^\mu$. Summarizing, the boundary of the world-sheet is located at $r = R^2/z_{\rm IR}$ and is a particular line constructed as follows (see figure 2):

- For each particle of momentum k^μ, draw a segment joining two points separated by $\Delta y^\mu = 2\pi k^\mu$.
- Concatenate the segments according to the insertions on the disk (corresponding to a particular color ordering).
- As gluons are massless, the segments will be light like. Due to momentum conservation, the diagram is closed.

The world-sheet, when expressed in T-dual coordinates, will then end in such a sequence of light-like segments (see figure 3) located at $r = R^2/z_{\rm IR}$.

Before T-dualizing, we can interpret the regulator D-brane as several $D3$-branes, one per each external particle, at a fixed radial distance and extended in the four directions $x^{1,3}$. The external states are then open strings ending in these branes, in such a way that two consecutive states end on each brane.

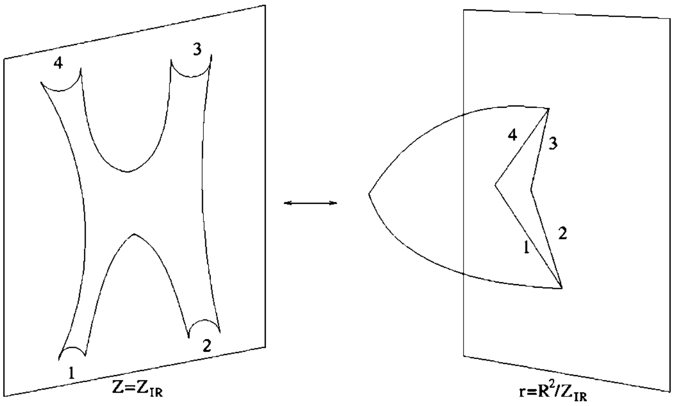

Figure 3. Comparison of the world sheet in original and T-dual coordinates. The hyperplane on the picture to the right should not be interpreted as a D-brane.

As we perform four T-dualities, these $D3$-branes become $D(-1)$ branes, or $D(-1)$-instantons, and the external states become open strings stretching a distance $\Delta y^\mu = 2\pi k^\mu$ between two such $D(-1)$-instantons. These instantons are located at the vertices, or cusps, of the above-mentioned polygons.

As we take away the IR cut-off, $z_{\mathrm{IR}} \to \infty$, the boundary of the world-sheet moves toward the boundary of the T-dual metric, at $r = 0$. At leading order in the strong coupling expansion, the computation that we are doing is formally the same as that we would do if we were computing the expectation value of a Wilson loop given by a sequence of light-like segments.

Our prescription is then that the leading exponential behavior of the n-point scattering amplitude is given by the area A of the minimal surface that ends on a sequence of light-like segments on the boundary:

$$\mathcal{A}_n \sim \mathrm{e}^{-\frac{\sqrt{\lambda}}{2\pi} A(k_1,\ldots,k_n)}. \tag{3.4}$$

Note that at strong coupling is more meaningful to talk about the geometric area of the world-sheet. This area is directly related to the log of the amplitude rather than to the amplitude itself. On the other hand, the BDS ansatz (2.7) is also an ansatz for the log of the amplitude[2]. Hence, the strong-coupling limit of the BDS ansatz can be directly compared to the geometrical area, times a proportionality factor, which is basically the cusp anomalous dimension at strong coupling.

We should stress that the strong-coupling computation is blind to the type or polarization of the external particles, and hence valid for a general scattering, MHV or non-MHV, among gluons or other particles, etc. Such information will contribute to prefactors in (3.4) and will be subleading in a $1/\sqrt{\lambda}$ expansion[3].

We have then reduced the problem of computing scattering amplitudes at strong coupling to the problem of finding minimal surfaces, or soup bubbles, in AdS. With the present technology, this turns out to be a very difficult problem; however, in the following we will show that such a surface can be found for the particular case of the scattering of four gluons.

[2] Note that the presence of non-iterating terms $\mathcal{O}(\epsilon)$ in the exponent of (2.7) allows only to write the log of the amplitude, and not the amplitude itself, up to terms of order $\mathcal{O}(\epsilon)$.

[3] In particular, such a prefactor could even vanish, as it should be the case if we were considering amplitudes of the form $\mathcal{A}(\pm, +, +, \cdots +)$.

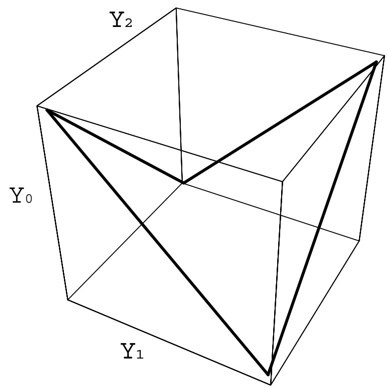

Figure 4. Polygon corresponding to the scattering of four gluons.

3.2. Scattering of four gluons

Consider the scattering of two particles into two particles, $k_1 + k_3 \to k_2 + k_4$ and define the usual Mandelstam variables

$$s = -(k_1 + k_2)^2, \qquad t = -(k_2 + k_3)^2. \tag{3.5}$$

According to our prescription we need to find the minimal surface ending in the following light-like polygon shown in figure 4.

As a warm up exercise it is instructive to consider the solution close to one of the cusps.

3.2.1. The single cusp solution. We start by considering the solution near the cusp where two of the light-like lines meet. So we consider two semi-infinite light-like lines meeting at a point. This case was already considered in [17]. The surface can be embedded into an AdS_3 subspace of the full AdS_5:

$$\mathrm{d}s^2 = \frac{-\mathrm{d}y_0^2 + \mathrm{d}y_1^2 + \mathrm{d}r^2}{r^2}. \tag{3.6}$$

We are interested in computing the surface ending on a light-like Wilson loop along $y^1 = \pm y^0$, $y^0 > 0$; see figure 5.

The problem has a boost and scaling symmetry that becomes explicit if we choose the following parametrization:

$$y_0 = \mathrm{e}^\tau \cosh\sigma, \qquad y_1 = \mathrm{e}^\tau \sinh\sigma, \qquad r = \mathrm{e}^\tau w. \tag{3.7}$$

Boosts and scaling transformations are simply shifts of σ and τ. The Nambu–Goto action becomes

$$S_{\mathrm{NG}} = \frac{R^2}{2\pi}\int \mathrm{d}\sigma\,\mathrm{d}\tau \frac{\sqrt{1-(w(\tau)+w'(\tau))^2}}{w(\tau)^2}. \tag{3.8}$$

One can explicitly check that $w(\tau) = \sqrt{2}$ solves the equations of motion and has the correct boundary conditions. Hence the surface is given by

$$r = \sqrt{2}\sqrt{y_0^2 - y_1^2}. \tag{3.9}$$

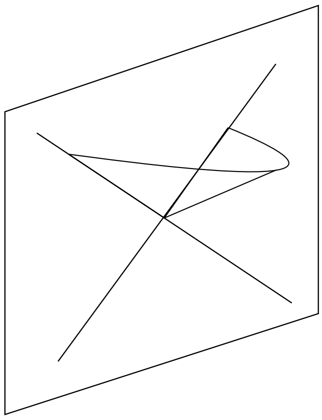

Figure 5. Single cusp solution.

3.2.2. Four cusps solution. Now, let us come back to the case of the solution with four cusps. In order to write the Nambu–Goto action it is convenient to use Poincaré coordinates (r, y_0, y_1, y_2), setting $y_3 = 0$ and parametrize the surface by its projection to the (y_1, y_2) plane. In this case, we obtain an action for two fields, r and t, living in the space parametrized by y_1 and y_2:

$$S = \frac{R^2}{2\pi} \int \mathrm{d}y_1\,\mathrm{d}y_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2}. \tag{3.10}$$

The classical equations of motion should then be supplemented by the appropriate boundary conditions. We consider first the case with $s = t$ where the projection of the Wilson lines is a square. By scale invariance we can choose the edges of the square to be at $y_1, y_2 = \pm 1$. The boundary conditions are then given by

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \qquad y_0(\pm 1, y_2) = \pm y_2, \qquad y_0(y_1, \pm 1) = \pm y_1. \tag{3.11}$$

Near each of the cusps the form of the solution should reduce to the single cusp solution (3.9). Making educated guesses satisfying the boundary conditions and with the correct properties near the cusps we propose

$$y_0(y_1, y_2) = y_1 y_2, \qquad r(y_1, y_2) = \sqrt{\left(1 - y_1^2\right)\left(1 - y_2^2\right)}. \tag{3.12}$$

Remarkably it turns out to be a solution of the equations of motion.

On dimensional grounds, the finite part of the amplitude is some function of the form $f(s/t)$. Hence, in order to capture the non-trivial dependence of the amplitude on the kinematics, we need to consider more general solutions with $s \neq t$. In this case the projection of the surface to the (y_1, y_2) plane will not be a square but a rhombus, with s and t given by the square of the distances between opposite vertices.

In order to find the solution for this more general case, it is instructive to study the surface (3.12) in terms of embedding coordinates. These are coordinates where we view AdS_5 as the following surface embedded in $R^{2,4}$:

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1. \tag{3.13}$$

The relation between embedding and Poincaré coordinates is

$$Y^\mu = \frac{y^\mu}{r}, \qquad \mu = 0, \ldots, 3$$
$$Y_{-1} + Y_4 = \frac{1}{r}, \qquad Y_1 + Y_4 = \frac{r^2 + y_\mu y^\mu}{r}. \tag{3.14}$$

The surface (3.12) is then given by

$$Y_0 Y_{-1} = Y_1 Y_2, \qquad Y_3 = Y_4 = 0. \tag{3.15}$$

Once we have written our solution in embedding coordinates, we note that we can apply $SO(2,4)$ transformations, that are linearly realized in this coordinates, in order to obtain new solutions. This $SO(2,4)$ symmetry is sometimes referred to as 'dual conformal symmetry' and should not be confused with the original $SO(2,4)$ symmetry associated with the original AdS space. It was first observed in computations at weak coupling in [18].

Solutions with $s \neq t$ can be obtained by starting from (3.12) and performing a boost in the $0-4$ direction:

$$Y_0 Y_{-1} = Y_1 Y_2, \qquad Y_4 = 0 \to Y_4 - v Y_0 = 0, \qquad \sqrt{1-v^2} Y_0 Y_{-1} = Y_1, Y_2. \tag{3.16}$$

Hence, we end up with two-parameter solutions, one related to the size of the initial square and another related to the boost parameter. The solution can be conveniently written as

$$r = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \qquad y_0 = \frac{a\sqrt{1+b^2} \sinh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \tag{3.17}$$

$$y_1 = \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \qquad y_2 = \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, \tag{3.18}$$

where we have written the surface as a solution to the equations of motion in conformal gauge:

$$\mathrm{i}S = -\frac{R^2}{2\pi} \int \mathcal{L} = -\frac{R^2}{2\pi} \int \mathrm{d}u_1 \, \mathrm{d}u_2 \frac{1}{2} \frac{(\partial r \partial r + \partial y_\mu \partial y^\mu)}{r^2}. \tag{3.19}$$

a and b encode the kinematical information of the scattering as follows:

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}, \qquad \frac{s}{t} = \frac{(1+b)^2}{(1-b)^2}. \tag{3.20}$$

According to the prescription, we should now plug the classical solution into the classical action in order to obtain the four-point scattering amplitude at strong coupling. However, in doing so, we obtain a divergent answer. That is of course the case, since we have taken the IR regulator away. In order to obtain a finite answer we need to reintroduce a regulator. A possibility would be setting the boundary conditions at some fixed value of $r = r_0$, and then take $r_0 \to 0$; however, it seems pretty hard to find classical solutions with such boundary conditions and it is more convenient to proceed as follows.

3.2.3. Dimensional regularization. Gauge theory amplitudes are regularized by considering the theory in $D = 4-2\epsilon$ dimensions. More precisely, one starts with $\mathcal{N} = 1$ in ten dimensions and then dimensionally reduces to $4 - 2\epsilon$ dimensions. For integer 2ϵ, this is precisely the low-energy theory living on a Dp-brane, where $p = 3 - 2\epsilon$. We regularize the amplitudes at strong coupling by considering the gravity dual of these theories. The string frame metric is

$$\mathrm{d}s^2 = f^{-1/2}\,\mathrm{d}x_{4-2\epsilon}^2 + f^{1/2}\left[\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{5+2\epsilon}^2\right], \qquad f = (4\pi^2 \mathrm{e}^\gamma)^\epsilon \Gamma(2+\epsilon)\mu^{2\epsilon} \frac{\lambda}{r^{4+2\epsilon}}. \tag{3.21}$$

Following the steps described above, we are led to the following action:

$$S = \frac{\sqrt{c_\epsilon \lambda}\mu^\epsilon}{2\pi}\int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}, \tag{3.22}$$

where $\mathcal{L}_{\epsilon=0}$ is the Lagrangian density for AdS_5. The presence of the factor r^ϵ will have two important effects. On one hand, previously divergent integrals will now converge (if $\epsilon < 0$). On the other hand, the equations of motion will now depend on ϵ, and we were not able to compute the full solution for arbitrary ϵ. However, we are interested in computing the amplitude only up to finite terms as we take $\epsilon \to 0$. In that case, it turns out to be sufficient to plug the original solution into the ϵ-deformed action[4]. After performing the integrals, we obtain

$$S \approx \sqrt{\lambda}\frac{\mu^\epsilon}{a^\epsilon}\,{}_2F_1\left(\frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2\right). \tag{3.23}$$

Expanding in powers of ϵ we get the final answer:

$$\mathcal{A} = e^{-\frac{\sqrt{\lambda}}{2\pi}A}, \qquad -\frac{\sqrt{\lambda}}{2\pi}A = iS_{\text{div}} + \frac{\sqrt{\lambda}}{8\pi}\left(\log\frac{s}{t}\right)^2 + \tilde{C}, \tag{3.24}$$

$$S_{\text{div}} = 2S_{\text{div},s} + 2S_{\text{div},t}, \tag{3.25}$$

$$iS_{\text{div},s} = -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}} - \frac{1}{\epsilon}\frac{1}{4\pi}(1-\log 2)\sqrt{\frac{\lambda\mu^{2\epsilon}}{(-s)^\epsilon}}. \tag{3.26}$$

This should be compared with the field theory expectations (2.7). We note that the general structure is in perfect agreement with the BDS ansatz. Once we use the strong-coupling behavior for the cusp anomalous dimension [19], $f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + \cdots$ we see that the leading divergence, as well as the finite piece, has the correct kinematical dependence with the correct overall coefficient.

3.3. A conformal Ward identity

An important ingredient of the previous computation was the existence of a dual $SO(2,4)$ symmetry, associated with the isometry group of the dual AdS_5 space. This symmetry allowed the construction of new solutions and fixed somehow the finite piece of the scattering amplitude. Naively, this conformal symmetry would imply that the amplitude is independent of s and t, since they can be sent to arbitrary values by a dual conformal transformation. The whole dependence on s and t arises due to the necessity of introducing an IR regulator. However, after keeping track of the dependence on the IR regulator, the amplitude is still determined by the dual conformal symmetry.

We can write a generic regularized area as follows:

$$A_n^{\text{reg}} = \text{Div} + \text{Fin}(x_i). \tag{3.27}$$

As already mentioned, the structure of IR divergences of scattering amplitudes is known to all loops. As a result, the divergent piece of the area is well known. Here x_i denote the position of the cusps. To a symmetry we associate a Ward identity that will impose certain constraints on scattering amplitudes. The dual conformal symmetry under consideration will impose the following relation for the finite part of the area [20–22]:

$$\sum_i \left(2x_i^\mu(x_i\cdot\partial_{x_i}) - x_i^2\partial_{x_i^\mu}\right)\text{Fin}(x_i) = \frac{f(\lambda)}{4}\sum_i \left(x_{i-1}^\mu + x_{i+1}^\mu - 2x_i^\mu\right)\log\left(x_{i-1,i+1}^2\right), \tag{3.28}$$

[4] Up to a contribution from the regions close to the cusps that add an unimportant additional constant term.

where $f(\lambda)$ is the value of the cusp anomalous dimension at strong coupling (properly normalized). Be aware that, as already mentioned, the area is directly related to the log of the amplitude, so the conformal Ward identity is simply (3.28) with no extra logs.

For the particular case of $n = 4$ and 5, the solution to this equation is actually unique!; hence, the finite piece of the scattering of four and five gluons is fixed in the case dual conformal symmetry persists to all values of the coupling constant.

The conformal Ward identity (3.28) was first written down in [20] and it was shown to hold to all orders in perturbation theory for the finite piece of the expectation value of light-like Wilson loops with cusps at x_i. As we will discuss in the following section, there are reasons to believe that MHV scattering amplitudes are dual to certain light-like Wilson loops, in which the dual conformal symmetry of scattering amplitudes acts as usual conformal symmetry. If this were the case, then dual conformal symmetry would be a symmetry of MHV scattering amplitudes to all values of the coupling constant.

3.4. Recent developments

In very interesting recent developments [23–25], see also [26], the dual conformal symmetry was shown to be part of a larger dual super-conformal symmetry.

As already mentioned, the supersymmetry of MSYM results in supersymmetric Ward identities. Besides ensuring the vanishing of certain class of amplitudes, these Ward identities relate also scattering amplitudes among different particles, very much like supersymmetry relates different components of the same super-multiplet.

It is then convenient to think of a gluon scattering amplitude as a component of a larger object, called the super amplitude[5]. In order to do that it is convenient to use the supersymmetric formalism introduced in [27]. To each particle in the spectrum of MSYM, with momentum k_i, one associates a pair of commuting spinors, such that $k_{\alpha,\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$, with $\alpha = 1, 2$ and $\dot\alpha = \dot 1, \dot 2$ and an anti-commuting variable, η^A, with $A = 1, \ldots, 4$ an $SU(4)$ index, hence $|p_i\rangle = \left|\lambda^i_\alpha, \tilde\lambda^i_{\dot\alpha}, \eta^A_i\right\rangle$. Note that the particle is guaranteed to be on shell, as $k_i^2 = 0$.

The supersymmetric scattering amplitude is then written in terms of such variables. When expanding in powers of η we recover scattering amplitudes among the different particles of MSYM. More precisely, a scattering process where the ith particle has helicity $h_i = 1 - n/2$ contains n powers of η_i.

Consider as an example the tree-level n-point MHV super amplitude, which can be written as

$$\mathcal{A}_n^{\rm MHV}(\lambda^1, \tilde\lambda^1, \eta_1, \ldots, \lambda^n, \tilde\lambda^n, \eta_n) \sim \frac{\delta^{(4)}\left(\sum_i \lambda^i_\alpha \tilde\lambda^i_{\dot\alpha}\right)\delta^{(8)}\left(\sum_i \lambda^i_\alpha \eta_i\right)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \tag{3.29}$$

where we have used the standard shorthand notation $\langle ij\rangle = \epsilon^{\alpha\beta}\lambda^i_\alpha\lambda^j_\beta$. The first delta function is the usual delta function of momentum conservation, and is due to the fact that the amplitude is translational invariant. In addition, the super amplitude is invariant under certain supersymmetries, which imply the second delta function. The usual gluon MHV amplitude (with minuses located at i and j) is then obtained by expanding in powers of η and keeping the term proportional to $(\eta_i)^4(\eta_j)^4$, we easily obtain

$$A_n^{\rm MHV}(ij) \sim \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \tag{3.30}$$

which agrees with the well-known result [27].

[5] To be more precise, a super amplitude groups together all amplitudes with a fixed number of external particles and fixed total helicity.

By writing the amplitudes in this form, in [23] it was shown that all the known amplitudes exhibit manifest dual super-conformal symmetry. This is a generalization of the statement that the gluon MHV scattering amplitude exhibits manifest dual conformal symmetry.

As a simple example, let us consider in more detail the tree-level amplitude (3.30). Dual super-conformal symmetry becomes more transparent after introducing dual coordinates

$$k_{i,\alpha\dot{\alpha}} = (x_i - x_{i+1})_{\alpha,\dot{\alpha}}, \qquad \eta_i^A \lambda_i^\alpha = \theta_i^{A\alpha} - \theta_{i+1}^{A\alpha}. \tag{3.31}$$

One can then regard the $x's$ as coordinates and the $\theta's$ as their super-partners. Note that these variables are not arbitrary, but they satisfy some constraints coming from the on-shell conditions,

$$(x_i - x_{i+1})^2 = 0, \qquad (\theta_i - \theta_{i+1})\lambda_i = 0. \tag{3.32}$$

On this constrained dual space it makes sense to consider super-conformal transformations. For instance, the inversion acts as [23]

$$x_{i,\alpha\dot{\beta}} \to \frac{x_{i,\beta\dot{\alpha}}}{x_i^2}, \qquad \theta_i^{A\alpha} \to \left(x_i^{-1}\right)^{\dot{\alpha}\beta} \theta_{i,\beta}^A. \tag{3.33}$$

It was then shown in [23] that the MHV super amplitude (3.30) transforms covariantly under inversions, namely

$$\mathcal{A}_n^{\mathrm{MHV}} \to \left(\prod_{i=1}^n x_i^2\right) \mathcal{A}_n^{\mathrm{MHV}}. \tag{3.34}$$

In a related development [24], see also [25], it was shown that the sigma model describing strings on $\mathrm{AdS}_5 \times S^5$ is invariant under a combination of the bosonic T-dualities described at the beginning of this section and novel fermionic T-dualities.

This 'fermionic T-duality' is a non-local redefinition of the fermionic world-sheet fields, very much like the redefinition of bosonic variables when we perform an ordinary T-duality. This duality can be applied to a generic supersymmetric background and maps it to another supersymmetric background with different *RR* fields and a different dilaton.

For the particular case of type IIB string theory on $\mathrm{AdS}_5 \times S^5$, certain combination of bosonic and fermionic T-dualities maps the model to itself! and, in particular, gluon scattering amplitudes are mapped to something very close to Wilson loops in the dual theory. Besides, the dual super-conformal symmetry of the original theory is mapped to the ordinary super-conformal symmetry of the dual model.

This would imply that dual conformal symmetry is indeed a symmetry of the scattering amplitudes (though there may be important issues related to the way dual super-conformal symmetry is broken when introducing a regulator.)

In spite of these developments, the proof of the duality between MHV scattering amplitudes and Wilson loops, to be described in the following section, remains an open problem.

4. Duality between scattering amplitudes and Wilson loops

According to the prescription described in the previous section, the computation of scattering amplitudes at strong coupling is mathematically equivalent to computing the expectation value of certain Wilson loops.

More precisely, one can consider the planar amplitude for n gluons, $A(k_1, \ldots, k_n)$, and an associated Wilson loop in position space, $W(x_1, \ldots, x_n)$, formed by light-like segments joining cusps at x_i, with $2\pi k_i = x_i - x_{i+1}$. The results of the previous section imply that both quantities are equivalent at strong coupling.

Quite remarkable, explicit computations [28, 29] show that this duality continues to hold also at weak coupling, provided we consider MHV scattering amplitudes. As already mentioned, the duality between amplitudes and Wilson loops would imply the dual conformal symmetry, since the dual conformal invariance becomes the ordinary conformal invariance of the Wilson loop computation.

The equivalence of both quantities at one loop can be stated as follows. The BDS ansatz (which to one loop is correct by construction) can be written as follows:

$$\log M_n = \mathrm{Div}_n + \frac{f(\lambda)}{4} a_1(k_1, \ldots, k_n) + h(\lambda) + nk(\lambda), \tag{4.1}$$

where a_1 is the one-loop amplitude, and $h(\lambda)$ and $k(\lambda)$ are independent on the kinematics and the number of gluons. On the other hand, we can consider the associated Wilson loop. As usual, given a contour C_n, the Wilson loop is defined as

$$\langle W_n \rangle = \frac{1}{N} \langle 0|\, \mathrm{Tr} P \exp\left(\mathrm{i}g \int_{C_n} A_\mu(x(\tau)) \dot{x}^\mu(\tau) \right) |0\rangle, \tag{4.2}$$

where $A = \mathrm{d}x^\mu A^a_\mu T^a$, T^a are the $SU(N)$ generators in the fundamental representation. For the present case, C_n is a polygonal contour, joining n cusps located at some points x_i, separated by light-like segments. Note that this is the same Wilson loop as that considered at strong coupling. The coupling to the scalars, usually present in BPS Wilson loops [30, 31], is not present in this case since the segments are light like. These Wilson loops are locally BPS, but the set of supersymmetries preserved by each segment are different.

Let us finally mention that such Wilson loops suffer from UV divergences, see for instance [28] and references therein, which can be regularized using dimensional regularization. The one-loop expectation value of the associated Wilson loop can be computed and has the general form,

$$\langle W_n \rangle = \tilde{\mathrm{Div}}_n + w_1(k_1, \ldots, k_n) + c(\lambda) + nd(\lambda), \tag{4.3}$$

where $c(\lambda)$ and $d(\lambda)$ are independent on the kinematics and the number of legs. Explicit computations show that $a_1 = w_1$.[6]

Note that the relation between scattering amplitudes to the expectation values of Wilson loops is subtle. For instance, the origin of the divergences in both computations is the opposite. Scattering amplitudes are IR divergent while Wilson loops are UV divergent. The duality is about the finite pieces.

Summarizing what we have said up to now:

- The BDS ansatz implies that the strong-coupling limit of MHV planar scattering amplitudes is given by the one-loop amplitude times the strong-coupling limit of the cusp anomalous. More precisely $a_{\mathrm{strong}}(k_1, \ldots, k_n) = f^{\mathrm{strong}} a_1(k_1, \ldots, k_n)$.
- The prescription developed in section 3 implies that scattering amplitudes and expectation values of Wilson loops agree at strong coupling, namely $a_{\mathrm{strong}}(k_1, \ldots, k_n) = w_{\mathrm{strong}}(k_1, \ldots, k_n)$.
- Explicit one-loop computations show that $a_1(k_1, \ldots, k_n) = w_1(k_1, \ldots, k_n)$.

Assuming BDS and using the second and third results, we arrive at

$$w_{\mathrm{strong}}(k_1, \ldots, k_n) = f^{\mathrm{strong}} w_1(k_1, \ldots, k_n). \tag{4.4}$$

Namely, the expectation value of a Wilson loop at strong coupling is given by its one-loop expectation value times the cusp anomalous dimension at strong coupling. Obviously, that is

[6] As an aside remark, let us mention that nonplanar corrections would spoil the conjectured relation between scattering amplitudes and Wilson loops. Related to this is the fact that nonplanar scattering amplitudes do not posses dual conformal symmetry [18].

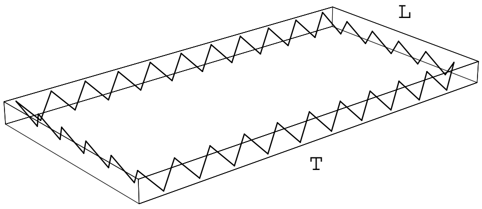

Figure 6. Zig-Zag configuration approaching the space-like rectangular Wilson loop.

a very non-trivial statement, and indeed, for the case of four and five edges, that is the case!. However, as we have seen, in this case the expectation value is fixed by symmetries. Hence, in order to test the BDS conjecture, we need to consider polygons with more than five edges.

Note that in order to disproof the BDS ansatz, it would be enough to find a single configuration for which (4.4) does not hold.

4.1. A test of the BDS conjecture

As just mentioned, in order to test the BDS conjecture, one would need to consider polygons of more than five edges. It seems very difficult to find explicit solutions for six edges. However, we can consider a zig-zag configuration with a large number of edges that approximates the rectangular Wilson loop see figure 5.

Such a Wilson loop is relevant when studying the quark–anti-quark potential at strong coupling, by means of the AdS/CFT duality [30, 31]. In the limit of very large T and L and for $T \gg L$, we can compute the expectation value both at weak and strong coupling obtaining

$$\log\left\langle W_{\text{rect}}^{\text{weak}}\right\rangle = \frac{\lambda}{8\pi}\frac{T}{L}, \log\left\langle W_{\text{rect}}^{\text{strong}}\right\rangle = \frac{\sqrt{\lambda}4\pi^2}{\Gamma(1/4)^4}\frac{T}{L}, \tag{4.5}$$

while the BDS ansatz prediction would be $\log\left\langle W_{\text{rect}}^{\text{strong}}\right\rangle = \frac{\sqrt{\lambda}}{4}\frac{T}{L}$. Hence, the BDS ansatz fails at strong coupling for a large enough number of gluons.

Note that the above surface contains a large number of cusps, and hence each of them will contribute with a divergent term. Such a contribution, however, grows like the perimeter $T + L \sim n$ and does not affect the coefficient in front of T/L.

The previous reasoning can also be repeated by considering the two loops amplitude versus the two loops Wilson loop expectation values. Explicit two loops computations for the rectangular Wilson loop show that at this order and for a large number of gluons, either BDS or the duality between scattering amplitudes and Wilson loops fails.

4.2. Six gluons

As explained above, the presence of the dual conformal symmetry fixes both the scattering amplitudes and the expectation values of the Wilson loops for the cases of $n = 4$ and $n = 5$, and in both cases, the result agrees with the BDS ansatz.

The BDS ansatz for six gluons satisfies the dual conformal Ward identities; however, it is not uniquely fixed by these. The general solution can always be written as the BDS ansatz plus an arbitrary function, usually called the remainder function, of the invariant cross ratios:

$$K.R(u_1, u_2, u_3) = 0 \to A_6 = A_{\text{BDS}} + R(u_1, u_2, u_3), \tag{4.6}$$

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}, \tag{4.7}$$

where K is, for instance, the generator of (dual) special conformal transformations, introduced in (3.28) . Note that these invariant cross ratios cannot be constructed for $n < 6$; this is the reason why the conformal Ward identity has a unique solution for $n = 4$ and 5.

Since for the case $n = 6$, the BDS ansatz is not anymore 'protected' by dual conformal symmetry, one would expect that it fails for such a case. A remarkable explicit computation for the scattering of six gluons at two loops [8] shows that indeed $R \neq 0$, and hence the BDS ansatz is to be modified for six gluons at two loops.

Regarding the duality between MHV scattering amplitudes and Wilson loops, a parallel computation for the two loops expectation value of the associated Wilson loops has also been carried out [32]. Quite remarkably, the duality between scattering amplitudes and Wilson loops continues to hold for this case! [33]. This is a strong indication that the duality may be true for any number of gluons at any loop order.

5. Conclusions

In this review we have described recent progress in computing planar scattering amplitudes on $\mathcal{N} = 4$ SYM at strong coupling by using the AdS/CFT correspondence. The computation reduces to a minimal surface problem in AdS, with boundary conditions fixed by the momenta of the external particles.

One of our main motivations was to test the BDS ansatz. Our results agree with this conjecture for $n = 4, 5$ but disagree for a large number of gluons. The agreement can be understood as due to the dual conformal symmetry. Moreover, explicit computations indeed show that the BDS ansatz is not correct for six gluons at two loops.

An important ingredient in the computation of amplitudes at strong coupling is the presence of a dual $SO(2, 4)$ conformal symmetry. This symmetry fixes the form of the amplitude for the case of four and five gluons. This symmetry is also observed at weak coupling at it is believe to be present for all values of the coupling constant. Furthermore, it is part of a larger dual super-conformal symmetry, unrelated to the original super-conformal symmetry of MSYM (but related to its higher nonlocal conserved charges [24, 25, 34]).

In addition, the strong coupling picture suggests a relation between amplitudes and Wilson loops, which seems to be a true relation and it survived several non-trivial checks. Such a relation would imply the existence of dual conformal symmetry for scattering amplitudes.

There are many directions one could try to follow:

- Find new solutions corresponding to the scattering of more than four gluons. Since we are ultimately interested in the regularized area, it may also be possible to calculate such an area without finding the explicit solutions.
- Try to use the machinery of integrability in order to find new constraints, very much like the Ward identity discussed in section 3, on the amplitudes.
- Try to prove/disprove the duality between Wilson loops and MHV scattering amplitudes. Despite some recent progress, this remains an open problem. The extension of such duality to the case of non-MHV amplitudes is also very interesting.
- Try to extend the set of ideas presented here to theories other than MSYM.

In any case, a bast structure behind scattering amplitudes and Wilson loops of MSYM is emerging and it seems clear that more is still to come!

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